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Radiation from Curved Dielectric Slabs and Fibers

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Abstract—**The form taken by the radiation condition in the local coordinate system, pertinent to the determination by perturbation methods of the radiation from curved radiating structures, is not the same as it is at very great distances. Specifically, it may contain a term that appears as if it were an incoming or growing wave. A detailed analysis is made of the appropriate form of the condition in cylindrical and toroidal systems, and is applied to the calculation of radiation from curved dielectric slabs and fibers.**

I. INTRODUCTION

WITH the introduction of glass fiber as a communication medium, it became necessary to understand and predict the radiation loss or leakage due to bending, with a view both to finding methods of reducing or preventing it, and to determining limits on bending for a given loss.

Marcatili [1] investigated the effects of bending of a dielectric slab in cylindrical coordinates using a method that was basically rigorous, though depending on mathematical asymptotic expansions at a later stage. The same method was applied to obtain an approximation to the leakage from a bent light guide of rectangular cross section. However, this process is not applicable to toroidal coordinates and the curved dielectric fiber.

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One method of treating radiation from curved structures is to attempt a perturbation analysis, treating the curvature as a small perturbation to the straight configuration. In so doing one is immediately concerned with matching fields at the structure surface, and hence, with the form of the radiation condition to be applied there. The difficulty arises from the fact that the perturbation analysis presents the field in local coordinate form and the range for which it is valid is limited to dimensions of the order of the size of the structure—in the present case, the bending radius. Since the radiation condition has to be applied at "infinity," i.e., at a distance much greater than the bending radius, the local coordinate form is quite useless. The radiation condition has to be inferred by an indirect process before it can be applied. It needs to be stressed that an "outward-looking" wave in the local coordinates is not necessarily (and in general, is not) the appropriate form to satisfy the radiation condition at very large distances. Thus the outgoing Hankel function $H_\nu^{(2)}(k\rho)$ behaves like an outgoing wave when $k\rho \gg \nu$, but looks more like a sum of a growing and an evanescent wave when $k\rho$ is small. To require only the evanescent term would be erroneous.

Since the cylindrical coordinate solution can, in any case, be set up rigorously, it might be asked what purpose is served by first using it to determine the radiation condition, and then applying this condition to a perturbation analysis. The advantage is threefold. It gives a better insight into what is going on, and may therefore indicate ways of controlling the radiation. Moreover, to get an analytic solution, asymptotic methods must in any case be applied, and this is readily done in the perturbation solution whose development depends on just such an expansion. And finally, it points the way to treating the far more difficult case of the optical fiber, for which no such solution exists. The natural coordinate system for the curved fiber would be a toroidal system, but no convenient solution of the Helmholtz equation exists there; neither is there any equivalent of the known asymptotic expansion of the Hankel function for that system.

We shall therefore briefly discuss the perturbation solution for the curved dielectric slab, using the rigorous formulation to get the radiation condition, and then proceed to a determination of the local form of the radiation condition in toroidal coordinates before solving the curved optical fiber case by a perturbation method.

II. CURVED DIELECTRIC SLAB—RADIATION CONDITIONS

Fig. 1 shows a dielectric slab of thickness b whose axis is bent to a radius of curvature R_0. The inner and outer radii are $R_1 = R_0 - b/2$ and $R_2 = R_0 + b/2$. A cylindrical coordinate system ρ,ϕ permits seeking solutions varying as $\exp(-j\nu\phi)$. Since $\phi = s/R_0$, where s is the axial coordinate, we can interpret $\nu/R_0 = k'$ as the propagation constant around the slab. A local coordinate y, measuring position across a transverse section, is defined by $\rho = R_0 + y$, so that field matching is performed at boundaries at $y = \pm b/2$.

It is important to assess the order of magnitude of ν. If k_0 is the propagation constant of the empty space outside the dielectric, and ϵ_r is the relative dielectric constant of the slab, then we expect k' to lie between k_0 and $k_1 = k_0\epsilon_r^{1/2}$, for the dominant mode. Since the field for $\rho > R_0 + b/2$ will involve $\exp(-j\nu\phi)H_\nu^{(2)}(k_0\rho)$, we see that, near the slab, $k_0\rho/\nu \sim k_0R_0/k'R_0$, which is of the order of, but less than, unity. Similarly, for $\rho < R_0 - b/2$ the field will involve $\exp(-j\nu\phi)J_\nu(k_0\rho)$. Inside the slab, both $J_\nu(k_1\rho)$ and $Y_\nu(k_1\rho)$ are utilized. Since R_0, and hence ν, will be large for small curvatures, we shall be interested in asymptotic expansions of Bessel functions with order of the same magnitude as the argument. Specifically, for $\rho > R_2$ the field will involve $H_\nu^{(2)} = J_\nu - jY_\nu$, while for $\rho < R_1$ the relevant form is J_ν only, since the field has to be finite at $\rho = 0$. Thus the field external to the slab takes different forms according to which surface is involved.

Debye's asymptotic expansions [2] can be put in the form

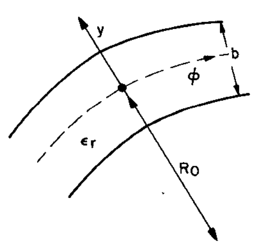


Fig. 1. Curved dielectric slab.

$$(2\pi\nu)^{1/2}J_\nu(\nu z) \sim \exp[-\nu f(z)](1 - z^2)^{-(1/4)} \quad (1)$$

$$(2\pi\nu)^{1/2}Y_\nu(\nu z) \sim -2\exp[+\nu f(z)](1 - z^2)^{-(1/4)} \quad (2)$$

where

$$f(z) = \tanh^{-1}(1 - z^2)^{1/2} - (1 - z^2)^{1/2} \quad (3)$$

and z is less than unity.

If we take $\nu z = k_0(R_0 + y)$, with $y \ll R_0$, we can expand $f(z)$ to first order about $z_0 = k_0R_0/\nu$ to obtain

$$-\nu f(z) = -\nu f(k_0R_0/\nu) + y(k'^2 - k_0^2)^{1/2}. \quad (4)$$

Hence, for y less than $-b/2$, where only J_ν is needed, the external field is represented solely by an evanescent wave, as might have been anticipated. But for $y > + b/2$, both J_ν and $-jY_\nu$ are needed to give $H_\nu^{(2)}$, and their relative proportions are seen to be given by the combination

$$\exp[-y(k'^2 - k_0^2)^{1/2}] + \frac{1}{2j}\exp[-2\nu f(k_0R_0/\nu)] \cdot\exp[+y(k'^2 - k_0^2)^{1/2}]. \quad (5)$$

Expression (5) is the equivalent of the radiation condition, an outgoing wave at infinity, in local coordinate form. In addition to the expected evanescent term there is a (small) growing term. Of course, (5) is only valid for a range of y limited by R_0, so the growing term does not grow indefinitely. But failure to include it in (5) necessarily leads to errors, and in fact leads to a prediction of no radiation at all! This is somewhat analogous to the use of a sinusoidal approximation to the current on a linear antenna, for which the examination of the corresponding input impedance leads to the conclusion that the arrangement is nonradiating. Nevertheless, by utilizing the Poynting vector at infinity, a sensible first-order approximation to the radiation can still be obtained, and Marcuse [3] has performed the equivalent calculation for the dielectric slab. Similarly to (5) he finds an asymptotic expression for the Hankel function, but limited to only the first decaying term; and he relates this field form to that in a straight slab. By comparison with known results the amplitude of the field for a given power flow is thus obtained. Then, by using the form of the Hankel function for very large arguments, the far field, and hence, the radiation from the slab, is deduced; and finally the attenuation is found from the radiation and power flow. This ingenious method does not require the growing component of (5). But unfortunately the process cannot be used for the curved fiber because of the absence of any known corresponding solution to the wave equation in toroidal coordinates.

III. CURVED DIELECTRIC SLAB—PERTURBATION SOLUTION

As already mentioned, a rigorous solution is possible in this case. If fields are derived from a magnetic component $H_z = \exp(-j\nu\phi)J_\nu(k_0\rho)$ for $\rho < R_1$, with corresponding forms for $R_1 < \rho < R_2$ and $\rho > R_2$, and the tangential field components are matched at $\rho = R_1$ and R_2, then the following equation, which is essentially an equation for ν, can be derived:

$$\frac{J_\nu(k_0R_1)J_\nu'(k_1R_1) - \epsilon_r^{1/2}J_\nu(k_1R_1)J_\nu'(k_0R_1)}{J_\nu(k_0R_1)Y_\nu'(k_1R_1) - \epsilon_r^{1/2}J_\nu'(k_0R_1)Y_\nu(k_1R_1)} = \frac{H_\nu^{(2)}(k_0R_2)J_\nu'(k_1R_2) - \epsilon_r^{1/2}J_\nu(k_1R_2)H_\nu^{(2)\prime}(k_0R_2)}{H_\nu^{(2)}(k_0R_2)Y_\nu'(k_1R_2) - \epsilon_r^{1/2}Y_\nu(k_1R_2)H_\nu^{(2)\prime}(k_0R_2)}. \quad (6)$$

The needed solutions are for ν large and complex. There is no known analytic solution to (6), though Marcatili used asymptotic expansions to obtain an approximate solution for small curvatures. Dang [4] has produced a computer program for solving (6) numerically, based on a calculation of the Bessel functions through a numerical evaluation of Hankel's integrals. Since ν will often involve only a small imaginary part, very high accuracy in the computations is necessary to calculate the attenuation accurately. The results can be compared to the perturbation solution, but differences may be difficult to interpret, due to the limited accuracy of the numerical computations.

To obtain a perturbation solution we put

$$H_z = \exp(-jk's)\, H = \exp(-jk's)[H_0 + H_1/R_0 + H_2/R_0^2 + \cdots]$$
$$k' = k_0'[1 + B_1 b/R_0 + \cdots] \quad (7)$$

and define

$$\gamma^2 = k_0^2\epsilon_r - k_0'^2$$
$$\delta^2 = k_0'^2 - k_0^2$$
$$\theta = b\gamma/2. \quad (8)$$

In local coordinate form the Helmholtz equation becomes

$$\left(1 + \frac{y}{R_0}\right)^2 \frac{\partial^2 H}{\partial y^2} + \frac{1}{R_0}\left(1 + \frac{y}{R_0}\right)\frac{\partial H}{\partial y} + \left[\left(1 + \frac{y}{R_0}\right)^2 k_0^2(\epsilon_r,1) - k'^2\right]H = 0 \quad (9)$$

where ϵ_r or 1 is taken, depending on whether or not the field is being considered inside or outside the slab.

For the zero-order solution we get the equation

$$\frac{\partial^2 H_0}{\partial y^2} + [k_0^2(\epsilon_r,1) - k_0'^2]H_0 = 0. \quad (10)$$

The solution, in the three regions, can be written

$$H_0 = (\cos\theta - A\sin\theta)\exp[\delta(y + b/2)], \qquad y < -b/2$$
$$= \cos\gamma y + A\sin\gamma y, \qquad -b/2 < y < b/2$$
$$= \frac{\cos\theta + A\sin\theta}{1 + K}\{\exp[-\delta(y - b/2)] + K\exp[\delta(y - b/2)]\}, \qquad y > b/2 \quad (11)$$

where, from (5),

$$K = -\tfrac{1}{2}j\exp\left\{\delta b - 2R_0\left[k_0'\log\frac{\delta + k_0'}{k_0} - \delta\right] - 2B_1k_0'b\log\frac{\delta + k_0'}{k_0}\right\}. \quad (12)$$

The term in K in the third expression of (11) represents the effect of the radiation condition, as applied to the dielectric slab; in its absence, the present method yields zero radiation. In the above forms, H_z has been matched at the two boundaries. The B_1 term appears in (12) because $\nu = k_0'(R_0 + bB_1)$, to first order, and the contribution is therefore $0(1)$ if B_1 itself has a significant magnitude.

The E_ϕ component is proportional to $(1/\epsilon)(\partial H_z/\partial y)$, and on matching at $y = \pm b/2$ we get two further equations, from which the constant A can be eliminated to yield the secular equation

$$\tan^2\theta + \tan\theta[(K - 1)\Delta + (K + 1)/\Delta] - 1 = 0 \quad (13)$$

with $\Delta = \epsilon_r\delta/\gamma$.

For small K the solution of (13) corresponding to the symmetrical mode in the straight slab is approximately given by $(1 + K)\tan\theta = \Delta$. This equation can be solved, for small K, in terms of the solution k_{00}' corresponding to $K = 0$. It gives $k_0' = k_{00}' + KL$ where

$$L = \frac{\epsilon_r}{\epsilon_r - 1}\cdot\frac{2\gamma_0^2\delta_0^2}{(2\epsilon_r k_0^2 + k_{00}'^2\delta_0 b)k_{00}'} \quad (14)$$

and γ_0 and δ_0 come from writing k_{00}' for k_0' in (8). In this relation k_{00}' is the solution obtained from

$$\tan(b\gamma_0/2) = \epsilon_r\delta_0/\gamma_0. \quad (15)$$

In calculating K it is sufficient, to this order of accuracy, to use δ_0 and γ_0 in (12).

In order to complete the solution we need B_1. As it happens, B_1 turns out to be neglible, and it can be ignored in (12), so that (14) does give k_0' without more ado. However, this feature is far from obvious, and there seems to be no way of demonstrating it other than solving the next higher order equation. This can be obtained from (9), from a consideration of the coefficient of $1/R_0$, in the form

$$\frac{\partial^2 H_1}{\partial y^2} + [k_0^2(\epsilon_r,1) - k_0'^2]H_1 = \frac{-\partial H_0}{\partial y} + k_0'^2 H_0(B_1 - 2y). \quad (16)$$

The solution can be written

$$H_1 = \cos\theta\exp[\delta(y + b/2)][C + y(B_1 + k_0^2/\delta k_0'^2) - y^2]k_0'^2/2\delta, \qquad y < -b/2$$
$$= -(k_0^2\epsilon_r/2\gamma^2)y\cos\gamma y + (k_0'^2/2\gamma)(D + B_1 y - y^2)\cdot\sin\gamma y, \qquad -b/2 < y < b/2$$
$$= (\cos\theta k_0'^2/2\delta)(\exp[-\delta(y - b/2)]\cdot\{y^2 - y(B_1 - k_0^2/\delta k_0'^2)] + F\{\exp[-\delta(y - b/2)] + K\exp[\delta(y - b/2)]\}), \qquad y > b/2 \quad (17)$$

with constants B_1, C, D, and F to be determined by matching H_z and E_ϕ at the two boundaries. Use has been made of the radiation condition in local form in determining the form of the terms in C and F.

The calculation of B_1 in this way is lengthy but straightforward, and gives $B_1 = KB$ where

$$B = \frac{\epsilon_r\delta^2[(1+X)(\epsilon_r^2 - X\gamma^2/\delta^2) + \epsilon_r^3\delta^2/k_0'^2]/4X}{X[(\epsilon_r-1)(\epsilon_r k_0'^2 - \gamma^2) + K\{\gamma^2 - \epsilon_r(\epsilon_r-1)k_0'^2\}] + \epsilon_r[k_0'^2(\epsilon_r-1) + K\gamma^2]} \tag{18}$$

and

$$X = \delta b(\epsilon_r\delta^2 + k_0')/2k_0^2.$$

Thus B_1 contains a factor K, and since K exponentially decreases with R_0, B_1 approaches zero for large R_0, and its effect on (12) is usually negligible. Since $\nu = k'R_0 = k_0'(R_0 + bB_1)$ to first order in $1/R_0$, we get, to first order in K and $1/R_0$,

$$k' = k_{00}' + K(L + Bk_{00}'b/R_0). \tag{19}$$

L and B are given by (14) and (18). To take the approximation further would be extremely tedious, but (19) should be accurate, even for quite modest values of R_0. Thus with $\lambda_0 = 1$ cm, $b = 0.4$ mm, $\epsilon_r = 4$, and $R_0 = 5b$ the value of ν is found as 21.06-j0.0001. The corresponding value from Dang's computation is 20.89-j0.00004. The real parts are quite close, but the imaginary parts, which give the attenuation, are too small to compare accurately. A closer agreement is obtained with a very leaky second-order mode with $b = 0.8$ mm and $R_0 = 2b$. The two values are, respectively, 11.628-j5.064 and 12.410-j5.139. For comparison, the first-order mode in this case has the imaginary parts of ν calculated as 8.10^{-5} and 5.10^{-6}, respectively.

IV. RADIATION CONDITION FOR A TOROIDAL STRUCTURE

Fig. 2 shows the toroidal geometry upon which three different coordinate systems have been erected. Spherical coordinates R,θ,ϕ are used in the investigation of the behavior at infinity. Cylindrical coordinates r,ϕ,z are used to construct an exact and suitable solution to the Helmholtz equation. Local coordinates are ρ,ψ,s and they approximate to cylindrical coordinates when the radius of the torus R_0 is very large. Note that the "axial" coordinate along the torus is $s = R_0\phi$, so that the relevant "axial" fields are E_ϕ and H_ϕ. The coordinates are interrelated by

$$z = R\cos\theta = \rho\sin\psi$$
$$r = R\sin\theta = R_0 + \rho\cos\psi. \tag{20}$$

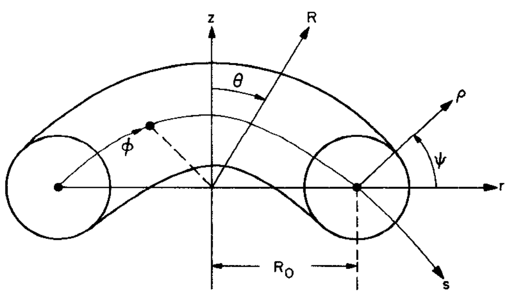


Fig. 2. Toroidal coordinates.

The space can be divided into the two regions $r < R_0$ and $r > R_0$. In the inner region the field must be finite at $r = 0$. In the outer region the field should look like an outgoing wave as $r \to \infty$. Outward going waves in the $\pm z$ direction imply sources at $z = 0$, so that forms like $\exp(-j\mu|z|)$ will be involved in setting up solutions. The outer region can therefore be considered by restricting ψ to the range 0 to $\pi/2$, since the symmetry due to the use of $|z|$ will cover the $-\pi/2$ to 0 range. Our aim is to develop a field which satisfies Maxwell's equations, behaves correctly at infinity and at the origin, and which can be developed on the torus to give a dominant term of the form $\exp(-jk's)K_m[k\rho(\gamma^2-1)^{1/2}]\cos m\psi$, this being the form of solution for a straight cylinder. Any "correction" terms to this form will be interpreted as the local structure of the radiation condition, in much the same way as was done for the extra term in (5).

We make the following definitions and observations:

- k propagation coefficient in the space outside the torus;
- k' propagation coefficient around the torus ($k' > k$ for a wave on a dielectric rod);
- $\nu = k'R_0$, giving $\nu\phi = k's$. As $R_0 \to \infty, \nu \to \infty$;
- $\gamma = k'/k > 1$, $\Gamma = (1 - 1/\gamma^2)^{1/2}$, $(\gamma^2-1)^{1/2} = |\gamma^2 - 1|^{1/2}\exp(-j\delta)$ with $\delta > 0$;
- $\beta = \sinh^{-1}(\gamma^2-1)^{-(1/2)}$, $\beta = \beta_r + j\beta_i$, giving $\tan\beta_i = \tan\delta\tanh\beta_r$, whence $\beta_i < \delta$.

From the form of the solution to the wave equation in cylindrical coordinates, we can build up a field which satisfies the Helmholtz equation and which is suitable to represent conditions in the toroidal geometry. After a considerable amount of searching the following structure was developed:

$$E_\phi = -\exp(-j\nu\phi)\int_{-\infty-j\delta}^{\infty+j\delta}\cosh mx \cdot\exp[-jk\rho(\gamma^2-1)^{1/2}\sin\psi\sinh x] \cdot kR_0\Gamma_x I_\nu(kR_0\Gamma_x)K_\nu'[k(R_0+\rho\cos\psi)\Gamma_x]\,dx, \quad 0 < \psi < \pi/2 \tag{21}$$

where

$$\Gamma_x = [(\gamma^2-1)\sinh^2 x - 1]^{1/2}.$$

This form is based on the spectral component $\exp(-j\nu\phi)\cdot\exp[-j\mu|z|]H_\nu^{(2)\prime}[r(k^2-\mu^2)^{1/2}]$, in which various changes of variables, including those in (20), have been made. The various functions that appear in (21) have been chosen to give an outgoing wave at infinity, and an appropriate field at the torus, as will be demonstrated in the ensuing paragraphs.

The contour of integration is shown in Fig. 3(a). The integrand possesses branch cuts at $\Gamma_x = 0$, or $x = \pm \beta$.

In order to demonstrate that (21) represents an outgoing wave, we transform to spherical coordinates, using (20), and take a new variable defined by $(\gamma^2 - 1)^{1/2} \sinh x = -j \sinh y$. Near the origin this gives $y \sim j(\gamma^2 - 1)^{1/2}x$. Since $\arg (\gamma^2 - 1)^{1/2} = -\delta$, we get $y = \pm(-\infty + j\pi/2)$ at $x = \pm(\infty + j\delta)$, and the contour for y becomes as shown in Fig. 3(b). The branch cuts are at $y = \pm j\pi/2$ and are not encountered. Equation (21) becomes

$$E_\phi = -\exp(-j\nu\phi) \int_{-\infty+j\pi/2}^{\infty-j\pi/2} \exp(-kR\cos\theta \sinh y) \cdot H_\nu^{(2)\prime}(kR\sin\theta\cosh y) J_\nu(kR_0\cosh y) \cdot [\pi k R_0 \cosh^2 y \cosh mx / 2(\gamma^2-1)^{1/2}\cosh x]\, dy. \tag{22}$$

For large R we can approximate the Hankel function by $\exp\{-j[kR\sin\theta\cosh y - \nu\pi/2 - \pi/4]\}(2/\pi kR \sin\theta \cosh y)^{1/2}$. The variable part of the total exponent is thus $-jkR[\cosh y \sin\theta - j\sinh y\cos\theta] = -jkR\cosh(y + j\theta - j\pi/2)$. Since $0 < \theta < \pi/2$ in the region considered, we can move the y contour to run horizontally through the point $j(\pi/2 - \theta)$, and on writing $y' = y + j\theta - j\pi/2$ we get the modified exponent $-jkR\cosh y'$. For large R the method of stationary phase gives its contribution from integration at $y' = 0$, and with the factor $R^{-(1/2)}$ already derived from the Hankel function we get a total variation with R of the form $\exp(-jkR)/R$. This verifies the outward-going character of (21) at infinity. It remains to find its form in the neighborhood of the torus.

Lemma: Let ν be large, z and b finite, and let a function $F_\nu(z)$ possess the asymptotic expansion, for large ν:

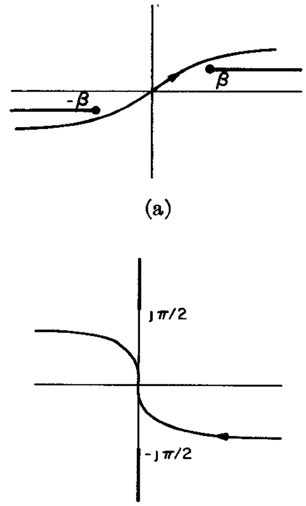
$$F_\nu(\nu z) = \exp[\nu f(z) + g(z)]\left\{1 + \frac{h(z)}{\nu} + 0\left(\frac{1}{\nu^2}\right)\right\}. \tag{23}$$

Then using Taylor's expansion we get

$$\frac{F_\nu(\nu z + bz)}{F_\nu(\nu z)} = \exp[bzf'(z)]\left\{1 + \frac{bzg'(z) + b^2z^2f''(z)/2}{\nu} + 0\left(\frac{1}{\nu^2}\right)\right\}. \tag{24}$$

Take $F_\nu(\nu z) = -(2\nu/\pi)^{1/2}K_\nu'(\nu z)$ and use Debye's expansion [2] for which

$$f(z) = \log\frac{1 + (1+z^2)^{1/2}}{z} - (1+z^2)^{1/2}$$



(a)

(b)

Fig. 3. (a) Contour of integration for x. (b) Contour of integration for y.

$$g(z) = (1/4)\log(1+z^2) - \log z.$$

Then

$$\frac{K_\nu'(\nu z + bz)}{K_\nu'(\nu z)} = \exp[-b(1+z^2)^{1/2}]\left\{1 + \frac{b}{2\nu}\left[\frac{1}{(1+z^2)^{1/2}} - \frac{b(2+z^2)}{1+z^2}\right] + 0\left(\frac{1}{\nu^2}\right)\right\}. \tag{25}$$

The term in $1/\nu$ is finite and approaches zero as $\nu \to \infty$. It will be ignored here, since we are only interested in results for large ν.

In a similar way we get

$$J_\nu'(\nu y + by)/J_\nu'(\nu y) \sim \exp[b(1-y^2)^{1/2}], \qquad y^2 < 1$$
$$Y_\nu'(\nu y + by)/Y_\nu'(\nu y) \sim \exp[-b(1-y^2)^{1/2}], \qquad y^2 < 1. \tag{26}$$

Also,

$$\frac{J_\nu'(\nu y)}{Y_\nu'(\nu y)} \sim \tfrac{1}{2}\exp[-2\nu f(y)] \tag{27}$$

where

$$f(y) = \tanh^{-1}(1-y^2)^{1/2} - (1-y^2)^{1/2}. \tag{28}$$

And finally we have

$$K_\nu'(\nu z)I_\nu(\nu z) \sim -1/2\nu z, \quad z > 0$$
$$Y_\nu'(\nu y)J_\nu(\nu y) \sim 1/\pi\nu y, \qquad y < 1. \tag{29}$$

It follows from these results that for z real we have

$$-\nu z I_\nu(\nu z)K_\nu'(\nu z + bz) \sim \tfrac{1}{2}\exp[-b(1+z^2)^{1/2}]. \tag{30}$$

But for z imaginary $= jy$, we use

$$K_\nu'(jy) = -\frac{\pi}{2}\exp(-j\nu\pi/2)H_\nu^{(2)\prime}(y)$$

and get

$$-\nu z I_\nu(\nu z)K_\nu'(\nu z + bz) \sim \tfrac{1}{2}\{\exp[-b(1-y^2)^{1/2}] + \tfrac{1}{2}j\exp[-2\nu f(y)]\exp[b(1-y^2)^{1/2}]\}. \quad (31)$$

It will be noticed that the first term is just the continuation of the function in (30) for imaginary z. The second term is *additional*, and only appears in that part of the range of z for which z is imaginary. As we shall see, its effect is to give inward or growing waves in the local coordinate system.

In (21), take $\nu z = kR_0\Gamma_x$, $bz = \rho\cos\psi\Gamma_x$. Then z is real for $x^2 > \beta^2$, and imaginary for $x^2 < \beta^2$. Applying (31) to (21) we get, for large R_0,

$$E_\phi = E_1 + E_2 \approx \tfrac{1}{2}\exp(-j\nu\phi)\int_{-\infty-j\delta}^{\infty+j\delta}\cosh mx \cdot\exp[-k\rho(\gamma^2-1)^{1/2}(\cosh x\cos\psi + j\sinh x\sin\psi)]\,dx + (j/4)\exp(-j\nu\phi)\cdot\int_{-\beta}^{\beta}\cosh mx\exp[k\rho(\gamma^2-1)^{1/2}(\cosh x\cos\psi - j\sinh x\sin\psi)]\exp(-2\nu f_x)\,dx \quad (32)$$

where

$$f_x = \tanh^{-1}(\Gamma\cosh x) - \Gamma\cosh x, \qquad \Gamma = (1-1/\gamma^2)^{1/2}.$$

E_1 can be evaluated by noting that the exponent can be written as $\cosh(x + j\psi)$, and if we take $x + j\psi = u$ the integral becomes

$$E_1 = \tfrac{1}{2}\exp(-j\nu\phi)\int_{-\infty+j(\psi-\delta)}^{\infty+j(\psi+\delta)}\cosh[m(u-j\psi)]\cdot\exp[-k\rho(\gamma^2-1)^{1/2}\cosh u]\,du. \quad (33)$$

Since $\arg(\gamma^2-1)^{1/2} = -\delta$, the integrand is convergent at both limits for $0 < \psi < \pi/2$. Moreover, (33) has no singularities so the contour can be displaced to the real u axis. Hence,

$$E_1 = \tfrac{1}{2}\exp(-j\nu\phi)\int_{-\infty}^{\infty}(\cosh mu\cos m\psi - j\sinh mu\cdot\sin m\psi)\exp[-k\rho(\gamma^2-1)^{1/2}\cosh u]\,du = \exp(-j\nu\phi)K_m[k\rho(\gamma^2-1)^{1/2}]\cos m\psi. \quad (34)$$

To within terms of order $1/\nu = (k'R_0)^{-1}$ this is the structure to be expected, and justifies the form chosen for the various factors used in setting up (21).

Had $j\sinh mx$ been used instead of $\cosh mx$, $\cos m\psi$ in (34) would have been replaced by $\sin m\psi$.

It remains to examine the form of E_2, and it is seen from (32) that the integrand is completely dominated by the term $\exp(-2\nu f_x)$, which has its greatest value when f_x is a minimum, which occurs at $x = 0$. Elsewhere it decreases rapidly, to zero at the limits. Hence, a saddle point calculation around $x = 0$ is sufficient and yields, on expanding the remaining exponent as a sum of modified Bessel functions:

$$E_2 \approx (j/4)\sum_0^\infty \epsilon_n I_n[k\rho(\gamma^2-1)^{1/2}]\cos n\psi \cdot\exp[-2\nu(\tanh^{-1}\Gamma - \Gamma)]\int_{-\infty}^{\infty}\cosh mx\cdot\exp\left(-\frac{\nu x^2\Gamma^3}{1-\Gamma^2}\right)dx \sim (j/4)\sum_0^\infty \epsilon_n I_n[k\rho(\gamma^2-1)^{1/2}]\cdot\cos n\psi\exp[-2\nu(\tanh^{-1}\Gamma - \Gamma)](\pi/kR_0)^{1/2}\cdot(\gamma^2-1)^{-(3/4)} \quad (35)$$

where $\epsilon_0 = 1$, $\epsilon_n = 2$ for $n > 0$.

If instead of $\cosh mx$ we had used $j\sinh mx$, we would have gotten $E_2 = 0$, to order $1/\nu$, for the corresponding integral.

Define the small quantity σ by

$$\sigma = (j/4)\exp\{-2kR_0[\gamma\tanh^{-1}(1-1/\gamma^2)^{1/2} - (\gamma^2-1)^{1/2}]\}\left(\frac{\pi}{kR_0}\right)^{1/2}(\gamma^2-1)^{-(3/4)}. \quad (36)$$

Then the local form of the field component E_ϕ (the same holds for H_ϕ) is

1) $\exp(-j\nu\phi)K_m[k\rho(\gamma^2-1)^{1/2}]\sin m\psi + \sigma 0(\nu^{-1})$

2) $\exp(-j\nu\phi)\{K_m[k\rho(\gamma^2-1)^{1/2}]\cos m\psi + \sigma\sum_0^\infty \epsilon_n I_n[k\rho(\gamma^2-1)^{1/2}]\cos n\psi\}. \quad (37)$

The radiation condition therefore gives, to this order of approximation, no additional term for modes odd in ψ. For modes symmetrical about $\psi = 0$ it is seen that the radiation condition involves not only an I_m term of magnitude $\sigma\epsilon_m$, but also a train of coupled modes; in much the same way as a propagating mode on a periodic structure is accompanied by a set of spatial harmonics, all propagating together with the phase of the main mode.

The field forms considered so far have been limited to $0 < \psi < \pi/2$. Inside the cylinder $r = R_0$ we require a form that is finite at the origin, and gives E_ϕ continuous at $\psi = \pi/2$. By inspection we get, as a dominant contribution,

$$E_\phi = -\exp(-j\nu\phi)\int_{-\infty-j\delta}^{\infty+j\delta}\cosh mx\cdot\exp[-jk\rho(\gamma^2-1)^{1/2}\sin\psi\sinh x - jm\pi]kR_0\Gamma_x\cdot\{I_\nu'[k(R_0+\rho\cos\psi)\Gamma_x]K_\nu'(kR_0\Gamma_x)\cdot I_\nu(kR_0\Gamma_x)/I_\nu'(kR_0\Gamma_x)\}\,dx, \qquad \pi/2 < \psi < \pi. \quad (38)$$

It is clear, from comparison with (21), that E_ϕ is continuous at $\psi = \pi/2$, since they both reduce to the same expression when m is even. When m is odd the continuity is not so obvious, and to demonstrate it, and also to show why the factor $\exp(-jm\pi)$ is included in (38), we consider the equivalent to (32) and (33). We get, analogously to (33),

$$E_1 = \tfrac{1}{2}\exp(-j\nu\phi)\int_{-\infty+j(\pi-\psi-\delta)}^{\infty+j(\pi-\psi+\delta)} \cosh[m(u-j\pi+j\psi)] \cdot \exp[-k\rho(\gamma^2-1)^{1/2}\cosh u - jm\pi]\,du. \quad (39)$$

Since, now, $(\pi-\psi)$ lies between 0 and $\pi/2$, we get the same result as (34); and this is what is required, since the full range of ψ must be covered. When m is odd the two fields are each zero at $\psi = \pi/2$, from (34), so that continuity is in any case maintained in this case too. The formula for E_2 becomes, for $r < R_0$,

$$E_2 \sim (j/4)\exp(-j\nu\phi)\int_{-\beta}^{\beta}\cosh mx \cdot \exp[k\rho(\gamma^2-1)^{1/2}[\cosh x\cos\psi - j\sinh x\sin\psi] - jm\pi]\exp(-2\nu f_x)\,dx. \quad (40)$$

This contribution comes via the term in K_ν' in (38), since it is replaced by the $H_\nu^{(2)\prime}$ form in this range. When combined with the exponent from the first I_ν' function, (40) is obtained, and agrees with (32) for m even. For m odd (40) is of the wrong sign, and hence, for continuity, a small additional term $\bar{E}_2$ needs to be added to (38). Its value is found to be

$$\bar{E}_2 = -j\tfrac{1}{2}(\cos m\pi - 1)\exp(-j\nu\phi)\int_{-\beta}^{\beta}\cosh mx \cdot \exp[-jk\rho(\gamma^2-1)^{1/2}\sin\psi\sinh x] \cdot \pi kR_0\bar{\Gamma}_x J_\nu'[k(R_0+\rho\cos\psi)\bar{\Gamma}_x]J_\nu(kR_0\bar{\Gamma}_x)\,dx \quad (41)$$

where

$$\bar{\Gamma}_x = [1-(\gamma^2-1)\sinh^2 x]^{1/2}.$$

Note that it is always possible to add a term of this character, since it is finite at the origin; hence, a unique radiation condition at the torus cannot be found from a consideration of fields in $r < R_0$. But for $r > R_0$ the radiation condition at infinity uniquely gives (32). The need for the term in (41) then follows in order to match at $\psi = \pi/2$. Its value is, of course, very small, since the asymptotic expansion of $J_\nu'J_\nu$ gives the factor $\exp(-2\nu f_x)$.

This concludes the derivation of the radiation condition for the toroidal geometry, contained essentially in (37).

V. THE BENT DIELECTRIC ROD OR OPTICAL FIBER

The analysis for a straight fiber has been given by, among others, Kao [5] and Clarricoats [6]. A central core, of size of the order of the wavelength, is surrounded by a cylindrical cladding of much greater diameter. The difference of dielectric constants is small, typical of the order of a percent or so, and the field is vanishingly small at the outer surface of the cladding. Although a refined calculation of the radiation due to bending would take into account both the core–cladding and the cladding–air interfaces, it seems probable that the major effect would occur at the former. We therefore take the cladding diameter infinite, and seek to apply the radiation condition (37) at the surface of the core.

Because of the form taken by (37) it is necessary to attend to the angular variable, and the resulting plane of polarization. Remembering that E_ϕ corresponds to the axial direction, we consider a field $E_s \propto \cos n(\psi+\alpha) = \mathrm{Re}\exp[-in(\psi+\alpha)]$ where i is used as a bicomplex variable[1] (distinct from j, which refers only to time variations; thus $i^2 = -1$, $j^2 = -1$, but $ij \neq -1$). Thus $\exp(-j\nu\phi)K_n[k\rho(\gamma^2-1)^{1/2}]\cos n(\psi+\alpha)$ will be replaced by $K_n\exp(-in\alpha)$, with an implied factor $\exp(-in\psi - i\nu\phi)$. However, because of the form of (37), we need first to write $\cos n(\psi+\alpha) = \cos n\psi\cos n\alpha - \sin n\psi\sin n\alpha$. The radiation condition provides a coupling (to first order) only to the cosine term, and, apart from the other modes which are entrained (their effect is to couple corresponding field components into the fiber), will introduce a term $I_n\cos n\psi\cos n\alpha$. More explicitly we can put the relevant field terms in the form

$$E_s = \exp(-j\nu\phi)\{K_n(\cos n\psi\cos n\alpha - \sin n\psi\sin n\alpha) + \sigma\epsilon_n I_n\cos n\psi\cos n\alpha\} \quad (42)$$

where the argument of the Bessel functions is $k\rho(\gamma^2-1)^{1/2}$. In complex form the right-hand side would be written as $[K_n + \sigma\cos n\alpha\exp(in\alpha)\,I_n]\exp(-in\alpha)$. Thus defining $\delta_{n\alpha}$ by

$$\delta_{n\alpha} = \sigma\epsilon_n\cos n\alpha\exp(in\alpha) \quad (43)$$

we can treat $\delta_{n\alpha}$ as the mode self-coupling coefficient for radiation. It is implied that, in this way of treating the problem, the (unspecified) amplitude of K_n, apart from the factor $\exp(-in\alpha)$, is real in i. Equation (42), via α, thus determines the orientation of the field relative to the plane of bending. The same interpretation applies, *mutatis mutandis*, inside the fiber.

Following Clarricoats, but with ρ,ψ,z in place of his r,θ,z, we can write, for the fields in the core,

$$E_s = a_nJ_n\exp(-in\alpha)$$
$$H_s = b_nJ_n\exp(-in\beta) \quad (44)$$

with an implied factor $\exp(-in\psi - jk's)$. Both a_n and b_n are real (in i). The argument of the Bessel functions is $K\rho$ where $K^2 = \omega^2\epsilon\mu_0 - k'^2$, and ϵ is the core permittivity. Two transverse components are needed for field matching, and they can be conveniently taken as H_ψ and H_ρ:

$$H_\psi = \frac{ijk'n}{K^2\rho}b_nJ_n\exp(-in\beta) - \frac{j\omega\epsilon a_n}{K}J_n'\exp(-in\alpha)$$

[1] A discussion on the use of bicomplex variables will be found in P. D. Crout, "The determination of antenna patterns of n-arm antennas by means of bicomplex functions," *IEEE Trans. Antennas Propagat.* (Commun.), vol. AP-18, pp. 686–689, Sept. 1970.

$$H_\rho = \frac{-jk'}{K} b_n J_n' \exp(-in\beta) - \frac{ijn\omega\epsilon}{K^2\rho} a_n J_n \exp(-in\alpha). \tag{45}$$

Outside the core, analogously to (44) and (45) we get

$$E_s = A_n(K_n + \delta_{n\alpha} I_n) \exp(-in\alpha)$$
$$H_s = B_n(K_n + \delta_{n\beta} I_n) \exp(-in\beta) \tag{46}$$

$$H_\psi = -\frac{ijk'n}{K_c^2\rho} B_n[K_n + \delta_{n\beta} I_n] \exp(-in\beta) + \frac{j\omega\epsilon_c}{K_c} A_n[K_n' + \delta_{n\alpha} I_n'] \exp(-in\alpha)$$

$$H_\rho = \frac{jk'}{K_c} B_n[K_n' + \delta_{n\beta} I_n'] \exp(-in\beta) + \frac{ij\omega\epsilon_c}{K_c^2\rho} A_n[K_n + \delta_{n\alpha} I_n] \exp(-in\alpha) \tag{47}$$

where the argument of the Bessel functions is $K_c\rho$, and A_n, B_n are real (in i). $K_c = (k'^2 - k_c^2)^{1/2}$, where $k_c^2 = \omega^2\epsilon_c\mu_0$ and ϵ_c is the cladding permittivity.

Equating E_s, H_s, H_ρ, and H_ψ at $\rho = a$, the core radius gives, after some reduction,

$$k'n\Lambda\left(\frac{1}{x^2} + \frac{1}{y^2}\right) = \exp[in(\beta - \alpha) - i(\pi/2)] \cdot \left[\frac{\omega\epsilon F_n}{x^2} + \frac{\omega\epsilon_c}{y^2} M_n\right] \tag{48}$$

$$k'\Lambda\left(\frac{F_n}{x^2} + \frac{\bar{M}_n}{y^2}\right) = \exp[in(\beta - \alpha) - i(\pi/2)] \cdot \left[\frac{\epsilon_c}{y^2} + \frac{\epsilon}{x^2}\right] n\omega \tag{49}$$

where $x = Ka$, $y = K_c a$, $\Lambda = b_n/a_n$ (real in i), and

$$F_n(x) = xJ_n'(x)/J_n(x)$$

$$M_n(y) = y\frac{K_n' + \delta_{n\alpha} I_n'}{K_n + \delta_{n\alpha} I_n} \sim y\frac{K_n'}{K_n}\left(1 + \frac{\delta_{n\alpha}}{yK_nK_n'}\right) \tag{50}$$

and $\bar{M}_n(y)$ is $M_n(y)$ with α replaced by β.

From (48), since Λ is real, $n\beta = n\alpha + m\pi + \pi/2$, whence it is seen that $\delta_{n\alpha} + \delta_{n\beta} = \epsilon_n\sigma$, independent of α (field orientation).

Eliminating Λ from (48) and (49), and considering the case $\epsilon \approx \epsilon_c$, so that $k' \approx k_c \approx k_1$, the resulting equation can be written

$$[F_n(x) + N_n(y)]^2 - \left[\frac{nk'}{k_c}\left(1 + \frac{x^2}{y^2}\right)\right]^2 \approx 0 \tag{51}$$

where

$$N_n(y) = \frac{yK_n'}{K_n}\left[1 + \frac{\epsilon_n\sigma}{yK_nK_n'}\right].$$

For HE_n modes this simplifies to

$$\frac{xJ_n(x)}{J_{n-1}(x)} = \frac{yK_n(y)}{K_{n-1}(y)}\left[1 + \frac{\epsilon_n\sigma}{yK_n(y)K_{n-1}(y)}\right] \tag{52}$$

where the term in σ gives the effect of the rod's curvature. It is interesting that it does not depend on the field orientation (angle α) with respect to the plane of bending.

Let x_0 and y_0 be the solution of (52) when $\sigma = 0$. Then $x^2 + y^2 = x_0^2 + y_0^2 = k_c^2a^2[(\epsilon/\epsilon_c) - 1]$, so if (52) has the solution $x = x_0 + \Delta x$, $y = y_0 + \Delta y$, then $x_0\Delta x + y_0\Delta y = 0$, and (52) gives, to first order,

$$\Delta x\left[\frac{J_n}{J_{n-1}} + \frac{x_0J_n'}{J_{n-1}} - \frac{x_0J_nJ_{n-1}'}{J_{n-1}^2}\right] = \frac{\epsilon_n\sigma}{K_{n-1}^2} + \Delta y\left[\frac{K_n}{K_{n-1}} + \frac{y_0K_n'}{K_{n-1}} - \frac{y_0K_nK_{n-1}'}{K_{n-1}^2}\right]. \tag{53}$$

Putting Δy in terms of Δx and solving gives

$$\Delta x = \frac{\epsilon_n\sigma x_0}{(x_0^2 + y_0^2)K_nK_{n-2}}. \tag{54}$$

In terms of a radiative correction $\Delta k'$ to k' we get, since $k' \approx k_1 \approx k_c$,

$$\frac{\Delta k'}{k_1} = \frac{-\sigma x_0^2\epsilon_n}{(k_1a)^4[(\epsilon/\epsilon_c) - 1]K_nK_{n-2}}. \tag{55}$$

(*Note:* $K_{n-2} = K_{2-n}$ for $n < 2$.) All terms in (55) are positive, so, since σ contains a factor j, (55) gives the attenuation.

Since $\gamma^2 - 1 = (k'^2 - k_1^2)/k_1^2 = y_0^2/k_1^2a^2 \ll 1$, the expression for σ can be approximated by

$$\sigma \approx (j/4)\exp[-\tfrac{2}{3}k_1R_0(y_0/k_1a)^3]\left(\frac{\pi}{k_1R_0(y_0/k_1a)^3}\right)^{1/2}. \tag{56}$$

For $n = 1$ and y_0 small, $K_1(y_0) \sim 1/y_0$ and (55) becomes

$$\left(\frac{\Delta k'}{k_1}\right)_{n=1} \approx -j\exp[-\tfrac{2}{3}k_1R_0(y_0/k_1a)^3]\left(\frac{\pi a}{R_0}\right)^{1/2} \cdot \frac{x_0^2y_0^{1/2}}{2(k_1a)^3[(\epsilon/\epsilon_c) - 1]}. \tag{57}$$

The nearest formula in the literature to compare this to appears to be Marcatili's [1] approximate calculation for a rectangular cross-section dielectric guide. It differs substantially in form, but the dominant part of the exponential is common to both. However, his approximation would not be expected to hold too well, (because of the omission of the external corner fields), if $\epsilon \approx \epsilon_c$, which is the basis of the above formulas. There is also no term $R_0^{-(1/2)}$ in his equation. However, the exponential term is so dominating in both formulas that, for weak radiation, the differences are not very significant.

As an example, (57) gives, for the attenuation per radian, for the case $\epsilon/\epsilon_c = 1.02$ and $a = \lambda$,

$R_0 = 1000\lambda$ attenuation = 0.4 neper/radian

$R_0 = 2000\lambda$ attenuation = 0.01 neper/radian.

These figures are of the same order of magnitude as given by Marcatali for the rectangular dielectric guide, but no closer comparison is possible.

APPENDIX

SOURCE CURRENTS FOR THE TOROIDAL FIELD

The sources to support this field are currents at $r = R_0$, mainly filamentary multipoles on the torus axis. Their form can be obtained by examining the difference between the two expressions for H_z with $r > R_0$ and $r < R_0$. In doing so we shall ignore the very small term in (41). From (21) and (38) we get, with $\zeta = (\mu/\epsilon)^{1/2}$,

$$\zeta H_z = -\exp(-j\nu\phi) \int_{-\infty-j\delta}^{\infty+j\delta} \cosh mx \cdot \exp[-jk\rho(\gamma^2-1)^{1/2} \sin\psi \sinh x] kR_0\Gamma_x^2 \cdot I_\nu(kR_0\Gamma_x) K_\nu[k(R_0+\rho\cos\psi)\Gamma_x]\, dx, \quad r > R_0 (\text{or } 0 < \psi < \pi/2) \tag{58}$$

$$\zeta H_z = -\exp(-j\nu\phi) \int_{-\infty-j\delta}^{\infty+j\delta} \cosh mx \cdot \exp[-jk\rho(\gamma^2-1)^{1/2} \sin\psi \sinh x - jm\pi] \cdot kR_0\Gamma_x^2 \{I_\nu[k(R_0+\rho\cos\psi)\Gamma_x] K_\nu'(kR_0\Gamma_x) \cdot I_\nu(kR_0\Gamma_x)/I_\nu'(kR_0\Gamma_x)\}\, dx, \quad r < R_0 (\text{or } \pi/2 < \psi < \pi). \tag{59}$$

Taking the difference of these expressions at $\psi = \pi/2$ gives the current density at $r = R_0$. Replacing ρ at $\psi = \pi/2$ by $|z|$ (this form also covers negative z), we get

$$\zeta I_\phi = -\exp(-j\nu\phi) \int_{-\infty-j\delta}^{\infty+j\delta} \cosh mx \cdot \exp[-jk|z|(\gamma^2-1)^{1/2} \sinh x] kR_0\Gamma_x^2 (I_\nu/I_\nu') \cdot [K_\nu I_\nu' - \exp(-jm\pi) I_\nu K_\nu']\, dx \tag{60}$$

where the argument of the Bessel functions is $kR_0\Gamma_x$. From the formula for the Wronskian, the expression in square brackets is $1/kR_0\Gamma_x$ when m is even. We shall examine this case first, also replacing I_ν/I_ν' by its asymptotic form (for large R_0) of $\Gamma_x \operatorname{sech} x/(\gamma^2-1)^{1/2}$:

$$\zeta I_\phi \sim -\exp(-j\nu\phi) \int_{-\infty-j\delta}^{\infty+j\delta} \cosh mx \Gamma_x^2 \operatorname{sech} x \cdot \exp[-jk|z|(\gamma^2-1)^{1/2} \sinh x](\gamma^2-1)^{-(1/2)}\, dx. \tag{61}$$

The factor $\cosh mx$, for m even, can be expanded in even powers of $\sinh x$. If we take $(\gamma^2-1)^{1/2} \sinh x$ as a new variable u, we have a remaining factor $\Gamma_x^2/(\gamma^2-1) \cosh^2 x = 1 - \gamma^2/(u^2+\gamma^2-1)$. To see how the formula behaves we first take the simplest case $m = 0$. Then

$$\zeta I_\phi = -\exp(-j\nu\phi) \int_{-\infty}^{\infty} \exp(-jk|z|u) \cdot \left[1 - \frac{\gamma^2}{u^2+\gamma^2-1}\right] du. \tag{62}$$

The integration of the first term gives $2\pi\delta(kz)$, i.e., a current filament on the torus axis. The remaining term gives a current of the form $-\pi\gamma^2(\gamma^2-1)^{-(1/2)} \cdot \exp[-k|z|(\gamma^2-1)^{1/2}]$, an attenuated current sheet at $r = R_0$. By multiple differentiations of these forms with respect to z we can get the corresponding forms when the initial factor $\cosh mx$ is retained, leading to the aforementioned current multipoles.

Although the multipole excitation was to be expected, the presence of the current sheet seems to call for an explanation. It is zero when $\gamma = 0$, although the formula is presumably no longer valid in this range. It is large when γ is near 1, and this could be associated with "end-fire" type radiation from the torus onto itself. It seems that, no matter how large the torus radius is, this self-illumination is present and affects the field distribution in the torus vicinity. The particular combination of filament and filamentary dipole which gives the factor $(2\gamma^2-3) + \cosh 2x$ requires no current sheet, and gives the main mode form $(2\gamma^2-3)K_0 + K_2 \cos 2\psi$. It is not immediately obvious what significance this combination may have.

When m is odd the two terms in square brackets in (60) cancel, to first order. Clearly, a z-directed multiplet is not needed in this case, and an examination of the discontinuity in H_r is required in the plane $z = 0$.

For $r > R_0$, H_r takes the form

$$\zeta H_r = \exp(-j\nu\phi) \operatorname{sgn}(z) \int_{-\infty-j\delta}^{\infty+j\delta} \cosh mx \sinh x \cdot \exp[-jk(\gamma^2-1)^{1/2}|z| \sinh x] \cdot kR_0\Gamma_x I_\nu(kR_0\Gamma_x) K_\nu'(kR_0\Gamma_x)(\gamma^2-1)^{1/2} \cdot \exp[-k(\gamma^2-1)^{1/2}(r-R_0)\cosh x]\, dx. \tag{63}$$

Because of the factor $\operatorname{sgn}(z)$, H_r is not continuous at $z = 0$ and the difference gives the current in the plane $z = 0$. However, at $z = 0$ the integrand is antisymmetrical and apparently gives zero on integration. The exception could be at $r = R_0$ when the exponential damping factor is zero. Moreover, since E_ϕ is continuous at $r = R_0$ and H_r comes from $\partial E_\phi/\partial z$, we can accommodate the region $r < R_0$ by writing $|r - R_0|$ instead of $(r - R_0)$ in (63). This clearly gives the correct exponent according to (38) when $\psi = \pi$. (The factor $\exp(-jm\pi)$ also needs to be included when $r < R_0$.)

Taking the limit for large R_0 we get

$$\zeta H_r = -\tfrac{1}{2}(\gamma^2 - 1)^{1/2} \exp(-j\nu\phi) \operatorname{sgn}(z) \cdot \int_{-\infty-j\delta}^{\infty+j\delta} \cosh mx \sinh x \cdot \exp\{-k(\gamma^2-1)^{1/2}[|r - R_0| \cosh x + j|z| \cdot \sinh x]\}\, dx \tag{64}$$

with an implied factor (-1) for $r < R_0$ when m is odd. We consider the case $m = 1$, for which the above factor can be written simply as sgn $(r - R_0)$. Then

$$\zeta H_r = (1/2k) \exp(-j\nu\phi) \operatorname{sgn}(z) \partial/\partial r \int_{-\infty-j\delta}^{\infty+j\delta} \sinh x \cdot \exp\{-k(\gamma^2-1)^{1/2}[|r - R_0| \cosh x + j|z| \cdot \sinh x]\}\, dx. \tag{65}$$

To find the value of the integral (which, by antisymmetry, is zero when $z = 0+$ except at $r = R_0$), integrate with respect to r over a region including R_0. Denoting the integral by J we get

$$k(\gamma^2 - 1)^{1/2} \int J dr = \int_{-\infty-j\delta}^{\infty+j\delta} \frac{\sinh x}{\cosh x} \exp[-jk|z|(\gamma^2-1)^{1/2} \sinh x]\, dx = -2j \int_0^{\infty+j\delta} \frac{\sinh x}{\cosh x} \sin[k|z|(\gamma^2-1)^{1/2} \sinh x]\, dx. \tag{66}$$

The integration can now be performed by taking $\sinh x$ as a new variable, and it gives

$$\int J\, dr = [\pi/jk(\gamma^2-1)^{1/2}] \exp[-k|z|(\gamma^2-1)^{1/2}]. \tag{67}$$

Since $J = 0$, except at $r = R_0$, we can therefore take

$$J = [\pi/jk(\gamma^2-1)^{1/2}]\delta(r - R_0) \tag{68}$$

where z has now been taken to zero. Hence, from the discontinuity in H_r at $z = 0$ we get

$$I_\phi = [j\pi/\zeta k^2(\gamma^2-1)^{1/2}] \exp(-j\nu\phi)\partial/\partial r[\delta(r - R_0)]. \tag{69}$$

This is a radially directed current doublet. Higher order multiplets come from further differentiations with respect to r, and correspond to larger values of m in (64).

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